Eur. J. Mech. B/Fluids, **17**, n° 4, 605–613, 1998

A new class of instabilities of rotating flows

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ABSTRACT. – The stability of flows which are the sum of a linear flow with circular or elliptic streamlines and a transverse standing wave is examined. A coordinate transformation annihilating the linear component of the flow is made, and the stability of the transformed flow is studied via two complementary methods. First, the stability with respect to local small scale perturbations is analyzed by virtue of the short wavelength stability method of Eckhoff and Lifschitz & Hameiri and it is found that all transformed flows are unstable with respect to such perturbations. Second, the corresponding linearized problem is studied via appropriately modified classical methods and global instabilities are found. It is shown that the growth rate of the global instabilities increases with their wavenumber and asymptotically approaches the value predicted by the short wavelength stability method. It is argued that the observed instabilities play an important role in transitions from laminar two-dimensional flows to turbulent three-dimensional ones. © Elsevier, Paris

Keywords. – Rotating fluids, Local instabilities, Global instabilities, Transition to turbulence.

1. Introduction

We examine the stability of flows which are the sum of a linear flow with circular or elliptic streamlines and a transverse standing wave. It is known that such flows are exact solutions of the Euler equations for an inviscid incompressible fluid (cf. Kelvin [1880], Chandrasekhar [1961], Craik and Criminale [1986], and Craik [1989]). These flows play an important role in the Rapid Distortion Theory (RDT) of Batchelor, Proudman, and Townsend (cf. Townsend [1976], Cambon *et al.* [1994]). Below we call them KRDT flows.

Although standing waves (which can be viewed as primary perturbations of the linear flows) have been extensively used for studying the stability of linear flows by Bayly [1986] and many others, the stability of composite flows did not receive enough attention until recently. In a series of papers by the present authors (cf. Lifschitz and Fabijonas [1996], Fabijonas *et al.* [1997], Fabijonas [1997], Miyazaki and Lifschitz [1997]) various particular aspects of the latter stability problem are considered. In the present paper (which is by necessity brief) we describe some novel techniques for studying the general stability problem and briefly discuss some of our observations. A detailed description of our findings will be given elsewhere. In contrast to our previous work, in the present paper we treat the most general linear flows by both geometrical optics and classical stability techniques.

We introduce a series of coordinate transformations which allows us to annihilate the linear component of the flows under consideration and make the KRDT flow to look as simple as possible. These transformations have a particularly simple form when the underlying linear flows have circular streamlines. Once the flows are simplified, we examine their stability via two complimentary methods. First, we study the stability with respect to localized short wavelength perturbations, which can be viewed as secondary perturbations of the

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corresponding trivial flows. By using the short wavelength stability theory of Eckhoff [1981], Lifschitz and Hameiri [1991], and Lifschitz [1994] we show that all the flows under consideration (even circular ones) are unstable with respect to short wavelength secondary perturbations. Second, we analyse the corresponding linearized problem via appropriately modified conventional methods and find global instabilities of the flows in question. We confirm the results of our asymptotic analysis and show that the growth rate of the global instabilities increases as their wavenumber increases and approaches the value predicted by the short wavelength analysis. It is very likely that the observed instabilities play an important role in transitions from laminar two-dimensional flows to turbulent three-dimensional ones.

2. Linear flows

Consider the standard Euler equations in coordinates $t_0, \mathbf{X}_0$,

$$\frac{\partial \mathbf{U}_0}{\partial t_0} + \langle \mathbf{U}_0, \nabla_0 \rangle \mathbf{U}_0 + \nabla_0 P_0 = 0, \qquad \langle \nabla_0, \mathbf{U}_0 \rangle = 0, \tag{1}$$

where $\mathbf{U}_0$ is the velocity and P_0 is the kinematic pressure. Here the notation $\langle \mathbf{A}, \mathbf{B} \rangle$ stands for the scalar product of two vectors $\mathbf{A}, \mathbf{B}$. This admits an exact solution of the form

$$\mathbf{U}_0^{eq} = \Omega \Big[-\frac{a_1}{a_2} X_{0,2}, \frac{a_2}{a_1} X_{0,1}, 0 \Big]^T, \quad P_0^{eq} = \frac{1}{2}\Omega^2 (X_{0,1}^2 + X_{0,2}^2), \tag{2}$$

where $a_1 \geq a_2$ without loss of generality. Here $A_{i,j}$ stands for the j component of a vector $\mathbf{A}_i$. This solution represents a plane-parallel flow with elliptical streamlines. We choose to non-dimensionalize (1) as follows:

$$\begin{aligned} &t_0 = t_1/\Omega, \qquad X_{0,i} = a_i X_{1,i}, \\ &U_{0,i} = \Omega a_i U_{1,i}, \quad i = 1, 2, 3, \qquad P_0 = \Omega^2 a_3^2 P_1, \end{aligned} \tag{3}$$

where $a_3 = [(a_1^2 + a_2^2)/2]^{\frac{1}{2}}$. Under this transformation of variables the Euler equations become

$$\frac{\partial \mathbf{U}_1}{\partial t_1} + \langle \mathbf{U}_1, \nabla_1 \rangle \mathbf{U}_1 + \mathcal{G}_1^{-1} \nabla_1 P_1 = 0 \qquad \langle \nabla_1, \mathbf{U}_1 \rangle = 0, \tag{4}$$

where $\mathcal{G}_1 = \mathcal{I} + \delta \mathrm{diag}[1, -1, 0]$ is the metric tensor of this new space, and δ is the eccentricity of the flow given by $\delta = (a_1^2 - a_2^2)/(a_1^2 + a_2^2)$, $0 \leq \delta < 1$. The flow (2) now takes the form

$$\mathbf{U}_1^{eq} = \Big[-X_{1,2}, X_{1,1}, 0 \Big]^T = \mathcal{J}_1 \mathbf{X}_1, \qquad P_1^{eq} = \frac{1}{2} \langle \mathcal{G}_1 \mathbf{X}_{1h}, \mathbf{X}_{1h} \rangle, \tag{5}$$

where $\mathcal{J}_1$ is the standard counter-clockwise rotation matrix in the horizontal $X_{1,1}, X_{1,2}$ plane, and $\mathbf{X}_{1h}$ denotes the horizontal projection of $\mathbf{X}_1$.

We now consider the same equations in a rotating coordinate system and write

$$\begin{aligned} &t_1 = t_2, \qquad \mathbf{X}_1 = \mathcal{S} \mathbf{X}_2, \\ &\mathbf{U}_1 = \mathcal{S}(\mathbf{U}_2 + \mathcal{J}_2 \mathbf{X}_2), \qquad P_1 = P_2 + \frac{1}{2} \langle \mathcal{G}_2 \mathbf{X}_{2h}, \mathbf{X}_{2h} \rangle, \end{aligned} \tag{6}$$

where $\mathcal{J}_2$ is the counter-clockwise rotation matrix in the $X_{2,1}, X_{2,2}$ plane, $\mathcal{S}(t_2)$ solves the equation $d\mathcal{S}/dt_2 = \mathcal{S} \mathcal{J}_2$, $\mathcal{S}(0) = \mathcal{I}$, and $\mathcal{G}_2 = \mathcal{S}^* \mathcal{G}_1 \mathcal{S}$ is the metric tensor in the rotating system. We use the Chain Rule in Eq. (4) to obtain

$$\frac{\partial \mathbf{U}_2}{\partial t_2} + \langle \mathbf{U}_2, \nabla_2 \rangle \mathbf{U}_2 + 2 \mathcal{J}_2 \mathbf{U}_2 + \mathcal{G}_2^{-1} \nabla P_2 = 0, \qquad \langle \nabla_2, \mathbf{U}_2 \rangle = 0, \tag{7}$$

where

$$\mathcal{J}_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{G}_2 = \mathcal{I} + \delta \begin{pmatrix} c_{2t_2} & -s_{2t_2} & 0 \\ -s_{2t_2} & -c_{2t_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Here and below c_τ, s_τ denote $\cos(\tau), \sin(\tau)$, respectively.

In this new coordinate system, the equilibrium solution is trivial by construction, $\mathbf{U}_2^{eq} = 0$, $P_2^{eq} = \text{const}$. Thus, we obtain the simplest expression of the linear flow (2) by carefully transforming the governing equations. We emphasize that for $\delta = 0$ the governing equations reduce to the standard Euler-Coriolis equations while for $\delta \neq 0$ they can be considered as the Euler-Coriolis equations in the space with time-dependent metric tensor.

3. KRDT modes

It is easy to show that (7) has the exact solution

$$\mathbf{U}_2^{eq} = \Upsilon \mathbf{A}_2(t_2) \sin\langle \mathbf{K}_2, \mathbf{X}_2 \rangle, \quad P_2^{eq} = \Upsilon \alpha_2(t_2) \cos\langle \mathbf{K}_2, \mathbf{X}_2 \rangle, \tag{9}$$

where $\mathbf{K}_2 = [\mathrm{s}_\theta, 0, \mathrm{c}_\theta]^T$ is the wave vector of the standing wave, $\mathbf{A}_2$ is its amplitude normalized in such a way that $\mathbf{A}_2(0) = [\mathrm{c}_\theta, 0, -\mathrm{s}_\theta]^T$, and Υ is the scaling factor. It can be shown that $\langle \mathbf{A}_2(t), \mathbf{K}_2 \rangle = 0$ by virtue of the incompressibility condition. The velocity $\mathbf{A}_2(t_2)$ and the pressure $\alpha_2(t_2)$ can be found from the equations

$$\begin{aligned} \frac{d\mathbf{A}_2}{dt_2} &= -2\mathcal{J}_2\mathbf{A}_2 + 2\frac{\langle \mathcal{J}_2\mathbf{A}_2, \mathbf{K}_2 \rangle}{\langle \mathbf{L}_2, \mathbf{K}_2 \rangle}\mathbf{L}_2, \quad \mathbf{A}_2(0) = [\mathrm{c}_\theta, 0, -\mathrm{s}_\theta]^T, \\ \alpha_2(t_2) &= 2\langle \mathcal{J}\mathbf{A}_2, \mathbf{k}_2 \rangle / \langle \mathbf{L}_2, \mathbf{K}_2 \rangle, \end{aligned} \tag{10}$$

where $\mathbf{L}_2 = \mathcal{G}_2^{-1}\mathbf{K}_2$ is the contravariant wave vector corresponding to the covariant wave vector $\mathbf{K}_2$. The equations for $\mathbf{A}_2$ have closed form solutions only for $\delta = 0$. In the latter case these were first examined by Kelvin [1880].

Following the same logic as before, we "adjust" the Euler equations (7) to obtain the simplest possible description of the flow. Namely, we consider a transformation of coordinates

$$t_2 = t_3, \quad \mathbf{X}_2 = \mathcal{R}\mathbf{X}_3, \quad \mathbf{U}_2 = \mathcal{R}\mathbf{U}_3, \quad P_2 = P_3, \tag{11}$$

where $\mathcal{R}$ is a rotation about the unit vector $\mathbf{e}_2$ such that $\mathbf{K}_2$ and $\mathbf{A}_2(0)$ turn into the unit vectors $\mathbf{e}_3$ and $\mathbf{e}_1$ of the rotated coordinate system, respectively. In the new coordinates, the Euler equations (7) assume the form

$$\frac{\partial \mathbf{U}_3}{\partial t_3} + \langle \mathbf{U}_3, \nabla_3 \rangle \mathbf{U}_3 + 2\mathcal{J}_3\mathbf{U}_3 + \mathcal{G}_3^{-1}\nabla_3 P_3 = 0, \quad \langle \nabla_3, \mathbf{U}_3 \rangle = 0, \tag{12}$$

where

$$\begin{aligned} \mathcal{J}_3 &= \begin{pmatrix} 0 & -\mathrm{c}_\theta & 0 \\ \mathrm{c}_\theta & 0 & \mathrm{s}_\theta \\ 0 & -\mathrm{s}_\theta & 0 \end{pmatrix}, \\ \mathcal{G}_3 &= \mathcal{I} + \delta \begin{pmatrix} \mathrm{c}_\theta^2 c_{2t_3} & -\mathrm{c}_\theta s_{2t_3} & \mathrm{s}_\theta \mathrm{c}_\theta c_{2t_3} \\ -\mathrm{c}_\theta s_{2t_3} & -c_{2t_3} & -\mathrm{s}_\theta s_{2t_3} \\ \mathrm{s}_\theta \mathrm{c}_\theta c_{2t_3} & -\mathrm{s}_\theta s_{2t_3} & \mathrm{s}_\theta^2 c_{2t_3} \end{pmatrix}, \end{aligned} \tag{13}$$

while, the standing wave in this new coordinate system has the form

(14) $$\mathbf{U}_3 = \Upsilon \mathbf{A}_3(t_3) \sin X_{3,3}, \qquad P_3 = \Upsilon \alpha_3(t_3) \cos X_{3,3},$$

where

$$\left[\frac{d}{dt_3} + \mathcal{M}\right] \begin{pmatrix} A_{3,1} \\ A_{3,2} \end{pmatrix} = 0, \quad \begin{pmatrix} A_{3,1}(0) \\ A_{3,2}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(15) $$A_{3,3} = 0, \qquad \alpha_3(t_3) = -2(1-\delta^2)\mathrm{s}_\theta A_{3,2}/(1 - \delta {\mathrm{s}_\theta}^2 c_{2t_3} - \delta^2 {\mathrm{c}_\theta}^2).$$

Here $\mathcal{M}$ is a 2 by 2 matrix with elements periodically depending on time (explicit expressions for these elements are omitted here, and in similar cases below, for the sake of brevity). Thus, the rotation $\mathcal{R}$ simplifies the form of the KRDT flow and removes its dependence on $X_{3,1}$ and $X_{3,2}$.

We note that $\mathbf{A}_3$ is the solution of a Floquet problem with periodic coefficients and therefore is itself either a periodic, quasi-periodic, or exponentially growing function. The KRDT flow may be viewed as the primary instability of the linear flow, i.e. the trivial solution. When $\delta = 0$,

(16) $$\mathbf{A}_3(t) = [\cos(2t_3\mathrm{c}_\theta), -\sin(2t_3\mathrm{c}_\theta), 0]^T, \quad \alpha_3(t) = 2\mathrm{s}_\theta \sin(2t_3\mathrm{c}_\theta).$$

In general, these quantities must be computed numerically. Periodic, quasi-periodic and growing solutions $\mathbf{A}_3$ are shown in Figures 1.

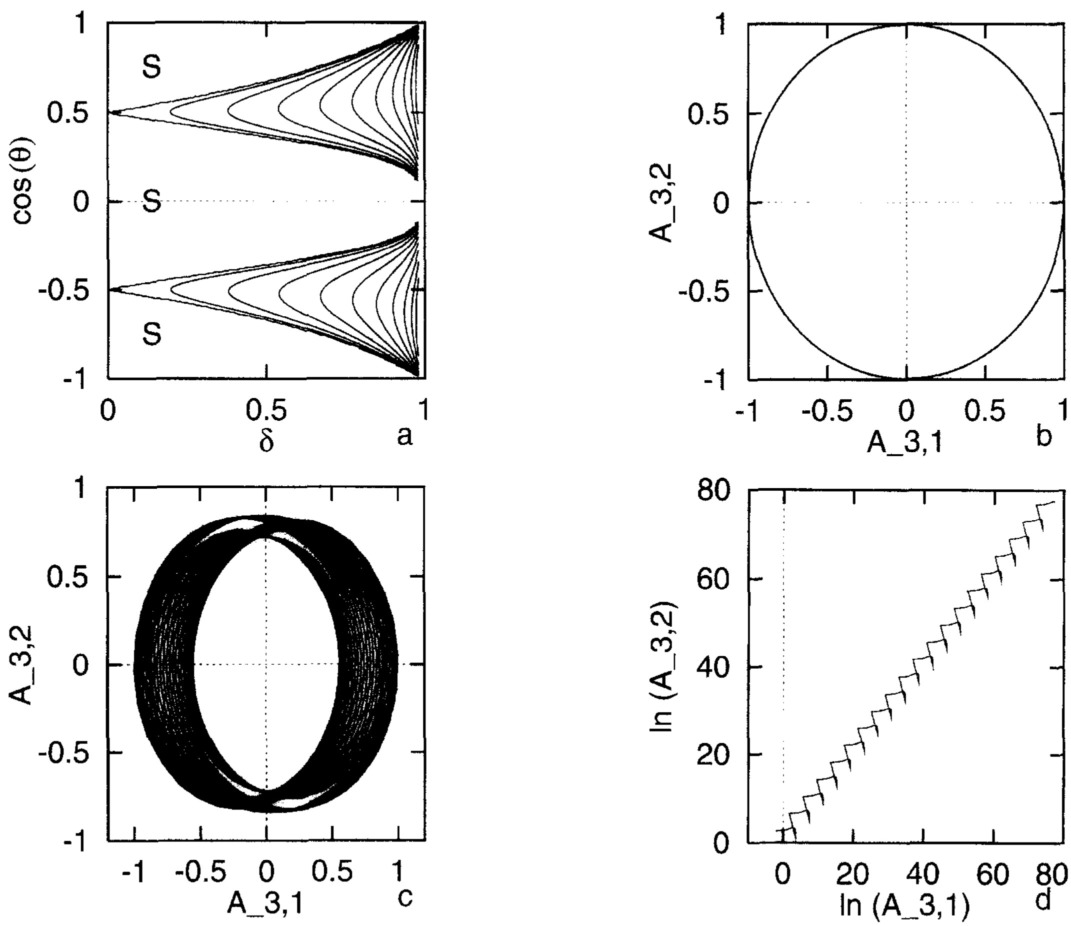

Fig. 1. – (a) The level lines of the growth rate of $\mathbf{A}_3$ in the (δ, c_θ) parameter plane. Regions labeled with 'S' have zero growth rate. (b-d) The phase plane of $\mathbf{A}_{3,h}(t)$ for $c_\theta = 0.8$ in the periodic (b: $\delta = 0$), quasi-periodic (c: $\delta = 0.5$), and exponentially increasing (d: $\delta = 0.9$) cases.

Since no further coordinate transformations are required below we suppress subscripts for the sake of brevity.

4. Perturbations

Let $\mathbf{U}^{eq}, P^{eq}$ be a particular KRDT flow. We perturb this solution by small perturbations

$$\mathbf{U} = \mathbf{U}^{eq} + \mathbf{u} \qquad P = P^{eq} + p, \tag{17}$$

and linearize (12) about this exact solution pair to obtain:

$$\frac{\partial \mathbf{u}}{\partial t} + \langle \mathbf{U}^{eq}, \nabla \rangle \mathbf{u} + (\mathcal{L} + 2\mathcal{J})\mathbf{u} + \mathcal{G}^{-1}\nabla p = 0, \quad \langle \nabla, \mathbf{u} \rangle = 0, \tag{18}$$

where

$$\mathcal{L} = \Upsilon \cos X_3 \begin{pmatrix} 0 & 0 & A_1(t) \\ 0 & 0 & A_2(t) \\ 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

is the velocity gradient matrix. We note that a perturbation of the KRDT flow can be viewed as a secondary perturbation of the linear flow.

5. Short wavelength stability

We analyze the stability of the KRDT flow in the short wavelenght limit via the method of Eckhoff [1981], Lifschitz and Hameiri [1991], and Lifschitz [1994]. We consider a perturbation in the form of localized rapidly oscillating wave envelope of the form

$$[\mathbf{u}(\mathbf{X}, t), p(\mathbf{X}, t)]^T = \exp[i\Phi(\mathbf{X}, t)/\epsilon][\mathbf{b}(\mathbf{X}, t), \epsilon\beta(\mathbf{X}, t)]^T, \tag{20}$$

with center $\mathbf{X}(t)$, phase $\Phi(\mathbf{X}, t)$, and amplitudes $\mathbf{b}(\mathbf{X}, t)$, $\beta(\mathbf{X}, t)$ vanishing outside a small vicinity of $\mathbf{X}(t)$. Here ϵ is a small parameter at our disposal characterizing the spatial scale of the perturbation. Denoting by $\boldsymbol{\xi}(\mathbf{X}, t)$ the wave vector of the wave packet, $\boldsymbol{\xi}(\mathbf{X}(t), t) = \nabla\Phi(\mathbf{X}(t), t)$, we can write a closed system of ordinary differential equations for $\mathbf{X}(t)$, $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(\mathbf{X}(t), t)$, $\mathbf{b}(t) = \mathbf{b}(\mathbf{X}(t), t)$, $\beta(t) = \beta(\mathbf{X}, t)$. This system has the form

$$\frac{d}{dt}\mathbf{X} - \mathbf{U}^{eq}(\mathbf{X}, t) = 0, \qquad \mathbf{X}(0) = \mathbf{X}_0, \tag{21}$$

$$\frac{d}{dt}\boldsymbol{\xi} + \mathcal{L}^T\boldsymbol{\xi} = 0, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \tag{22}$$

$$\frac{d}{dt}\mathbf{b} + (\mathcal{L} + 2\mathcal{J})\mathbf{b} + i\beta\boldsymbol{\chi} = 0, \;\; \langle \mathbf{b}, \boldsymbol{\xi} \rangle = 0, \;\; \mathbf{b}(0) = \mathbf{b}_0, \tag{23}$$

where $\boldsymbol{\chi} = \mathcal{G}^{-1}\boldsymbol{\xi}$ is the contravariant wave vector. It describes the behavior of the wave packet's to leading order. Taking the scalar product of the first equation in (23) with $\boldsymbol{\chi}$ and using equation (22) and the second equation in (24) we express β in terms of $\mathbf{b}$ as follows,

$$i\beta = -2\langle (\mathcal{L} + \mathcal{J})\mathbf{b}, \boldsymbol{\xi} \rangle / \langle \boldsymbol{\chi}, \boldsymbol{\xi} \rangle. \tag{24}$$

Substituting this expression in the first equation in (23) we obtain the closed form equation for b alone,

$$\frac{d}{dt}\mathbf{b} + (\mathcal{L} + 2\mathcal{J})\mathbf{b} - 2\frac{\langle(\mathcal{L} + \mathcal{J})\mathbf{b}, \boldsymbol{\xi}\rangle}{\langle\boldsymbol{\chi}, \boldsymbol{\xi}\rangle}\boldsymbol{\chi} = 0, \qquad \mathbf{b}(0) = \mathbf{b}_0. \tag{25}$$

The main result of the short wavelength stability theory is that the underlying flow is unstable provided that for a certain choice of $\mathbf{X}_0$, $\boldsymbol{\xi}_0$, $\mathbf{b}_0$ equation (25) has a growing solution. It is clear from equation (21) that the origin is a fixed point of the system and we choose to examine the flow there. Equation (22) yields

$$\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3 - \Upsilon\int_0^t \zeta dt]^T, \tag{26}$$

where $\boldsymbol{\xi}_0 = [\xi_1, \xi_2, \xi_3]^T$, and $\zeta = \xi_1 A_1 + \xi_2 A_2$. Eliminating b_3 by virtue of the incompressibility condition,

$$b_3 = -(\xi_1 b_1 + \xi_2 b_2)/(\xi_3 - \Upsilon\int_0^t \zeta dt), \tag{27}$$

where, for simplicity, we assume that the denominator does not vanish, and substituting the result in equation (25) we obtain the closed form equations for b_1, b_2 which can be written as

$$\left[\frac{d}{dt} + \mathcal{M}\right]\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0, \qquad \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{28}$$

Here $\mathcal{M}$ is a 2 by 2 matrix with time-dependent elements which are either periodic, or quasi-periodic, or exponentially growing. We numerically simulate (28) using appropriate techniques for solving two-dimensional systems of equations with periodic, quasi-periodic and growing coefficients, and find the instability regions for various parameter values. For two representative choices of δ and θ these regions are shown in Figures 2. These figures clearly demonstrate that for the particular choices of parameter values instabilities are always present. This fact is true in general as well. The details of the computation are rather involved and will be presented elsewhere. Particular cases are considered in Lifschitz and Fabijonas [1996], Fabijonas *et al.* [1997], and Fabijonas [1997].

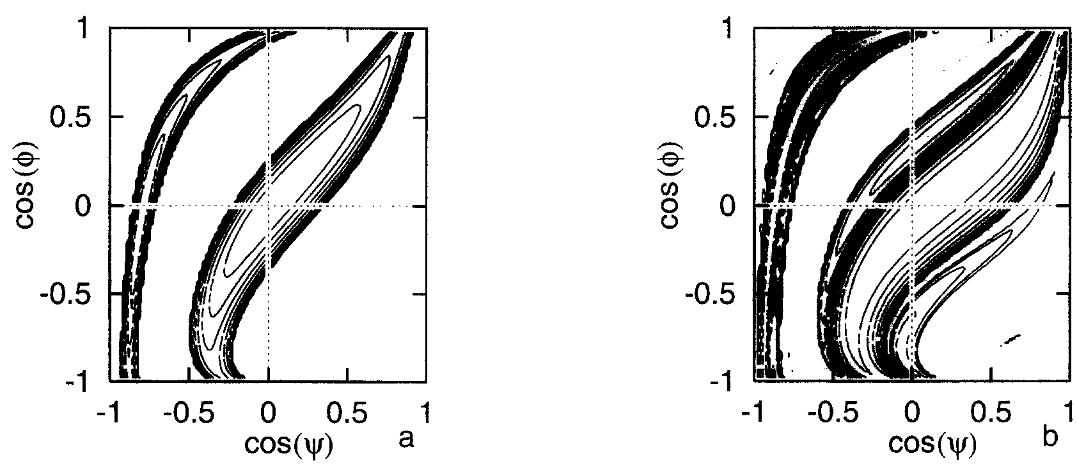

Fig. 2. – Level lines of the growth rates of the secondary perturbations $\mathbf{b}(t)$ in the (c_ϕ, c_ψ) parameter plane, where $[\xi_1, \xi_2, \xi_3]^T = [s_\phi c_\psi, s_\phi s_\psi, c_\phi]^T$ for $c_\theta = 0.8$ and (a) $\delta = 0$ with a maximum of 0.31 at (0.08,0.10), and (b) $\delta = 0.5$ with a maximum of 0.35 at (-0.08,-0.88).

6. Classical statibility

In this section, we consider a more general perturbations of the KRDT flow and expand them in Fourier series as follows:

$$[u_1, u_2, u_3, p]^T = \sum_{-\infty<m<\infty} \exp(i\langle \boldsymbol{\xi}_m, \mathbf{X}\rangle)[u_{m,1}, u_{m,2}, u_{m,3}, p_m]^T, \tag{29}$$

where $\boldsymbol{\xi}_m = [\xi_1, \xi_2, m + \xi_3]^T$ is the real valued wave vector which is independent of time, and $[u_{m,1}, u_{m,2}, u_{m,3}, p_m]^T$ is the complex valued amplitude which depends only on time. The constants ξ_1, ξ_2, $-\infty < \xi_i < \infty$ denote the wavenumbers in the X_1, X_2 directions, respectively, and ξ_3, $0 \le \xi_3 \le 1$ is the Floquet modulation exponent. We assume hereinafter that $\xi_3 \neq 0$. The analysis for the case $\xi_3 = 0$ is analogous. For convenience we denote by $\chi_m = [\chi_{m,1}, \chi_{m,2}, \chi_{m,3}]^T$ the contravariant wave vector, $\chi_m = \mathcal{G}^{-1}\boldsymbol{\xi}_m$.

Substituting expression (29) in equation (18) we obtain the following block-three-diagonal equations for the amplitude

$$\left[\frac{d}{dt} + \begin{pmatrix} 0 & -2c_\theta & 0 & \chi_{m,1} \\ 2c_\theta & 0 & 2s_\theta & \chi_{m,2} \\ 0 & -2s_\theta & 0 & \chi_{m,3} \\ \xi_1 & \xi_2 & \xi_3 + m & 0 \end{pmatrix}\right]\begin{pmatrix} u_{m,1} \\ u_{m,2} \\ u_{m,3} \\ ip_m \end{pmatrix} \tag{30}$$

$$+\frac{\Upsilon}{2}\begin{pmatrix} \eta & 0 & A_1 & 0 \\ 0 & \eta & A_2 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{m-1,1} \\ u_{m-1,2} \\ u_{m-1,3} \\ 0 \end{pmatrix} + \frac{\Upsilon}{2}\begin{pmatrix} -\eta & 0 & A_1 & 0 \\ 0 & -\eta & A_2 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{m+1,1} \\ u_{m+1,2} \\ u_{m+1,3} \\ 0 \end{pmatrix} = 0,$$

where $\eta = \xi_1 A_1 + \xi_2 A_2$. Taking the scalar product of the first three equations (30) with the vector $\boldsymbol{\xi}_m$ allows us to solve for the pressure in terms of the other variables:

$$\begin{aligned} ip_m = [&-2\xi_2(c_\theta u_{m,1} + s_\theta u_{m,3}) + 2(\xi_1 c_\theta + \xi_2 s_\theta)u_{m,2} \\ &- \Upsilon\eta(u_{m-1,3} + u_{m+1,3})]/\langle \chi_m, \boldsymbol{\xi}_m\rangle. \end{aligned} \tag{31}$$

Furthermore, we can use the incompressibility condition (18b) to reduce the dimension of the system by one:

$$u_{m,3} = -(\xi_1 u_{m,1} + \xi_2 u_{m,2})/(\xi_3 + m). \tag{32}$$

Substituting expressions (31), (32) into equation (30) we obtain the following reduced block-three-diagonal equation for $u_{m,1}, u_{m,2}$ only

$$\left[\frac{d}{dt} + \mathcal{M}_m^{(0)}\right]\begin{pmatrix} u_{m,1} \\ u_{m,2} \end{pmatrix} + \mathcal{M}_m^{(-)}\begin{pmatrix} u_{m-1,1} \\ u_{m-1,2} \end{pmatrix} + \mathcal{M}_m^{(+)}\begin{pmatrix} u_{m+1,1} \\ u_{m+1,2} \end{pmatrix} = 0, \tag{33}$$

where $\mathcal{M}_m$ are 2 by 2 matrices with time-dependent elements. We numerically integrate these equations using the Floquet (cf., e.g., Yakubovich and Starzhinskii [1976], and Miyazaki and Lifschitz [1997]) and local Lyapunov exponent (cf., e.g., Ott [1993]) techniques and find the growth rates as functions of ξ_1, ξ_2 for various parameter values. For two representative parameter sets the growth rates are given in Figures 3. These figures show that the classical growth rates are always greater than zero and tend to their short wavelength values

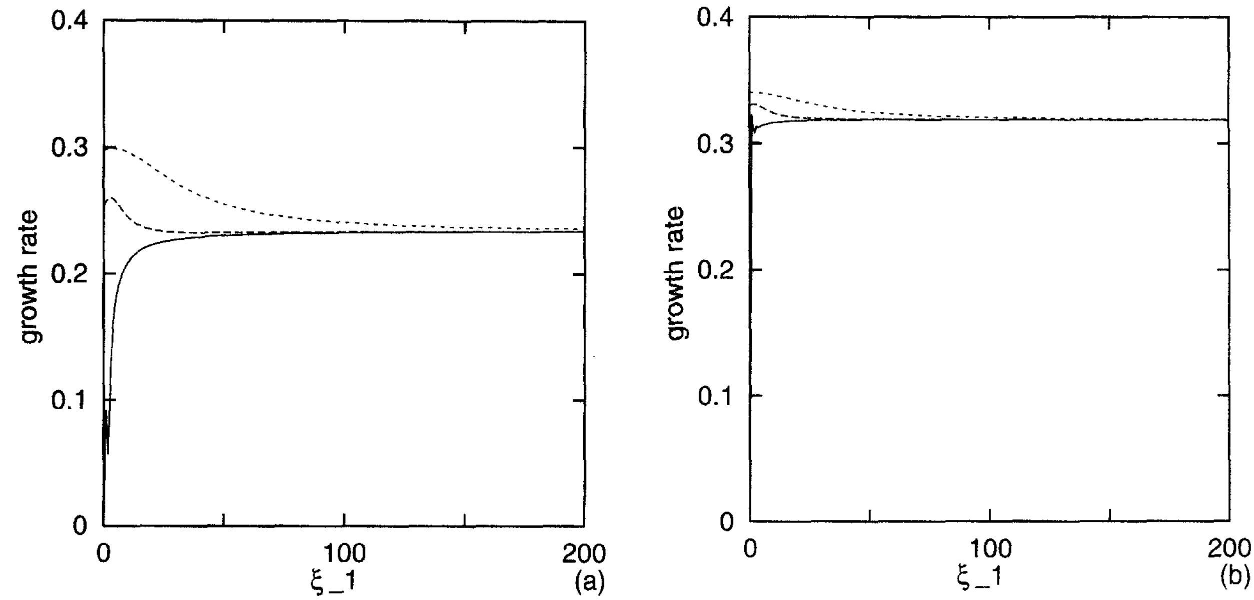

Fig. 3. – The growth rate of the general perturbations as a function of ξ_1 for $\xi_3 = 0.5$ and $\xi_2 = 0, 10, 40$ (from bottom to top, respectively) for $c_\theta = 0.8$ and (a) $\delta = 0$, and (b) $\delta = 0.5$.

when $|\xi_1|, |\xi_2| \to \infty$. We emphasize that these limiting values are smaller than the maximum values shown in Figures 2 because the orientation of the corresponding vectors $\boldsymbol{\xi}$ is not optimal.

7. Conclusions

In the present paper we propose a novel technique for studying the stability of flows which are the sum of an elliptical flow and a transverse standing wave. We demonstrate that in an appropriate rotating coordinate system the elliptical part of the flow can be completely eliminated. Once this is done we apply some modern techniques for studying the stability of inhomogeneous flows and show that all the flows in question are strongly unstable. In other words, the transverse standing waves which can be considered as primary instabilities of elliptical flows are themselves strongly unstable. This observation suggests that plane-parallel elliptical (and even circular) flows are prone to much stronger instabilities than previously anticipated and are likely to be destroyed due to instability mechanisms different from the ones which are conventionally used to explain their behaviour (cf., e.g., Bayly *et al.* [1988]).

The work of the first author was partially supported by NSF grant DMS-9623033. Computer resources were provided by the Laboratory for Advanced Computing at the University of Illinois at Chicago throught NSF grant number NSCP-CDA94l3928.

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(Received 27 June 1997,
revised 17 February 1998,
accepted 20 February 1998)